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The γ-Poincaré quantum group from quantum group contraction

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This paper is dedicated to the memory of my friend Ansgar Schnizer

Abstract. We propose a contraction of the de Sitter quantum group leading to a Poincaré quantum group in any dimensions. The method relies on the coaction of the de Sitter quantum group on a non-commutative space, and the deformation parameter q is sent to one. The bicrossproduct structure of this γ-Poincaré quantum group is exhibited and shown to be dual to the one of the κ-Poincaré Hopf algebra, at least in two dimensions.

1. Introduction

In the realm of Hopf algebras, several propositions for a deformed enveloping algebra of the D-dimensional Poincaré algebra $U_\kappa(\mathcal{P}(D))$ have been made in the recent past [1–4]. In these approaches, the basic tool is a contraction of the deformed enveloping algebra $U_q(so(D+1))$, *in which the deformation parameter q is simultaneously sent to its classical* value one. One particular feature of these deformations is that they are minimal, in the sense that the commutation relations are only slightly modified, but not too minimal since they are no longer cocommutative. Furthermore, these deformations are physically interesting since they involve a dimensional parameter κ which sets a scale in the theory that could in principle be determined by some measurement.

In two dimensions, another contraction of $U_q(su(2))$, where q is left unaffected, leads to a different deformation of the Poincaré enveloping algebra [5]. This Hopf algebra turns out to be dual to Woronowicz's construction of a Poincaré quantum group $E_q(2)$ [6].

Regarding the Poincaré quantum group, a construction has been made in [7], similar to the classical one where the Poincaré group is the inhomogeneous group associated with the Lorentz group. This procedure is based on a deformation of the Lorentz group, and requires the introduction of dilatations. It was later shown to be a semidirect (co)-product based on a braided quantum group structure [8]. In the present γ-Poincaré quantum group, the Lorentz subgroup is classical and there are no dilatations.

Here we are mainly interested in the dual of the κ-Poincaré algebra. Mathematically, these Hopf algebras are deformations based on non-semisimple Lie algebras. When the Lie algebra is simple, there is a natural dual Hopf algebra, conventionally known as the algebra of functions on the quantum group, and the R-matrix provides the elegant link between these dual structures [9]. At the present time, there is no known R-matrix for the deformed Poincaré algebra (except in dimension three [1]), therefore the investigation of the dual along this line is impossible. A potentially fruitful approach is to use the fact that the κ-Poincaré algebra is an example of a bicrossproduct of Hopf algebras, as was recently shown in [10].

Possible dual structures have been proposed by different authors, principally obtained by the quantization of the Poisson bracket on the algebra of functions on the classical group [3, 4, 11].

In this paper, we first extend a previous construction [12], initially developed for a two-dimensional spacetime, in which a Poincaré quantum group is obtained by a contraction of the corresponding de Sitter quantum group (sections 2 and 3). The deformation parameter q is sent to one as well, and similarly a dimensional parameter γ enters the final Hopf algebra. From a duality point of view, this is a natural starting point since the de Sitter quantum group and the deformed enveloping de Sitter algebra are known to be dual.

Next, by unravelling the γ-Poincaré quantum group bicrossproduct structure, we provide a strong hint that this it is actually dual to the κ-Poincaré algebra (sections 4 and 5). Finally in two dimensions, we are able to show that these bicrossproducts are precisely dual to each other (section 6). This provides an alternative duality proof to the one in [13]. Two appendices give some technical details used in the main text.

2. The Hopf algebra $Fun(SO_q(N, \mathbb{R}))$

The complex orthogonal quantum group is defined in [9] as the non-commutative algebra with unity and generators $T = (t_{ij}), i, j = 1, \ldots, N$, subject to the relation $\mathcal{R}_t T_1 T_2 = T_2 T_1 \mathcal{R}_t$, where the R-matrix is

$$\mathcal{R}_t = q \sum_{i \neq i'}^{N} e_{ii} \otimes e_{ii} + \sum_{i,j; i \neq j, j'}^{N} e_{ii} \otimes e_{jj} + q^{-1} \sum_{i \neq i'}^{N} e_{i'i'} \otimes e_{ii} + (q - q^{-1}) \sum_{i>j}^{N} e_{ij} \otimes e_{ji}$$
$$- (q - q^{-1}) \sum_{i>j}^{N} q^{\rho_i - \rho_j} e_{ij} \otimes e_{i'j'} \overset{\text{odd}}{+} e_{\frac{N+1}{2}, \frac{N+1}{2}} \otimes e_{\frac{N+1}{2}, \frac{N+1}{2}}. \tag{2.1}$$

Here $\overset{\text{odd}}{+}$ means that the term is present only for odd N. We use the notation $i' = N + 1 - i$, the integer part $M = [\frac{N-1}{2}]$ and the numbers ρ_i, for $1 \leqslant i \leqslant M$,

$$\rho_i = \frac{N}{2} - i \qquad \rho_{i'} = -\rho_i \qquad \rho_{M+1} = 0 \qquad \text{(for odd } N\text{)}.$$

The orthogonality conditions are

$$TCT^T C^{-1} = CT^T C^{-1} T = \mathbb{1} \qquad \text{with} \qquad C = \sum_{i=1}^{N} q^{\rho_i} e_{i'i}.$$

The complete Hopf algebra structure is specified by the homomorphisms

$$\Delta(T) = T \dot{\otimes} T \qquad \epsilon(T) = \mathbb{1} \qquad S(T) = CT^T C^{-1}. \tag{2.2}$$

The quantum N-dimensional complex space $O_q^N(\mathbb{C})$ is defined as the non-commutative algebra with unity generated by the N elements x_i subject to the relation

$$f(\hat{\mathcal{R}}_t)(x \otimes x) = 0 \qquad \text{with} \qquad f(t) = \frac{t^2 - (q + q^{1-N})t + q^{2-N}}{q^{-1} + q^{1-N}} \tag{2.3}$$

and $\hat{\mathcal{R}}_t = P\mathcal{R}_t$ is the permuted R-matrix. There is a coaction of the quantum group on the quantum space given by

$$\delta(x) = T\dot{\otimes}x \tag{2.4}$$

which preserves the quadratic form $x^T Cx$.

The quantum group real form we are considering here is specified by the anti-involution

$$T^* = DC^T T(C^{-1})^T D^{-1} \tag{2.5}$$

where $D = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_N)$, with $\epsilon_i^2 = 1, \epsilon_{i'} = \epsilon_i$ for $i = 1, \ldots, N$, and $\epsilon_i = 1$ for $i = i'$. These ϵ's represent in a way the signature of the quadratic form in the quantum space, and characterize the real quantum algebra $Fun(SO_q(N, \mathbb{R}; \epsilon_i))$. Similarly the quantum space is turned to a quantum real space $O_q^N(\mathbb{R})$ with the help of the anti-involution $x^* = DC^T x$.

For our geometric construction, it is more convenient to choose a real set of generators for the quantum space, $z_i = M_{ij}x_j = z_i{}^*$, with the matrix and its inverse

$$\begin{aligned} M &= \frac{1}{\sqrt{2}} \sum_{i=1}^{N} (\alpha_i e_{ii} + \beta_i e_{i'i}) \\ M^{-1} &= \frac{1}{\sqrt{2}} \sum_{i=1}^{N} (\gamma_i e_{ii} + \delta_i e_{i'i}) \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} &(\alpha_1, \ldots, \alpha_M) = (1, \ldots, 1) \qquad (\alpha_{M'}, \ldots, \alpha_N) = (-\mathrm{i}\epsilon_M q^{\rho_M}, \ldots, -\mathrm{i}\epsilon_1 q^{\rho_1}) \\ &\beta_j = \mathrm{i}\alpha_j \qquad \gamma_j = \frac{1}{\alpha_j} \qquad \delta_j = \frac{1}{\beta_{j'}} \qquad \text{for } j \neq j' \\ &\alpha_{\frac{N+1}{2}} = \beta_{\frac{N+1}{2}} = \gamma_{\frac{N+1}{2}} = \delta_{\frac{N+1}{2}} = \frac{1}{\sqrt{2}}. \end{aligned}$$

Accordingly, we take new real generators $V = (v_{ij}) = MTM^{-1}$ for the algebra $Fun(SO_q(N, \mathbb{R}))$, which satisfy slightly different orthogonality conditions

$$\begin{aligned} V\widehat{C}V^T &= \widehat{C} \qquad \text{with} \qquad \widehat{C} = MCM^T \\ V^T\widetilde{C}V &= \widetilde{C} \qquad \text{with} \qquad \widetilde{C} = M^{-1T}CM^{-1}. \end{aligned} \tag{2.7}$$

The comultiplication and counit are similar to (2.2), but the antipode is now

$$S(V) = \widehat{C}V^T\widetilde{C}. \tag{2.8}$$

In this real basis, the quantum space relations (2.3) become

$$\begin{aligned} &z_i z_j - q z_j z_i - z_{i'} z_{j'} + q z_{j'} z_{i'} = \mathrm{i}(z_{i'} z_j - q z_j z_{i'} + z_i z_{j'} - q z_{j'} z_i) \qquad i < j, i < i', j < j' \\ &z_{j'} z_{i'} - q z_{i'} z_{j'} - z_j z_i + q z_i z_j = -\mathrm{i}(z_{j'} z_i - q z_i z_{j'} + z_j z_{i'} - q z_{i'} z_j) \qquad i < j, i > i', j < j' \\ &z_{i'} z_j - q z_j z_{i'} + z_i z_{j'} - q z_{j'} z_i = -\mathrm{i}(z_i z_j - q z_j z_i - z_{i'} z_{j'} + q z_{j'} z_{i'}) \qquad i < j, i < i', j > j' \\ &\epsilon_i [z_i, z_{i'}] = \mathrm{i}\frac{q^2 - 1}{q^2 + 1} \sum_{k=i+1}^{M} \left(\frac{1 + q^2}{2}\right)^{k-i} \epsilon_k (z_k^2 + z_{k'}^2) \overset{\text{odd}}{+} \mathrm{i}\frac{q^2 - 1}{q + 1} \left(\frac{1 + q^2}{2}\right)^{M-i} z_{\frac{N+1}{2}}^2 \end{aligned} \tag{2.9}$$

and the quadratic form is diagonal

$$z^T\widetilde{C}z = \frac{1+q^{2-N}}{1+q^2}\sum_{k=1}^{M}\left(\frac{1+q^2}{2}\right)^k \epsilon_k(z_k^2+z_{k'}^2) \overset{\text{odd}}{+} \frac{1+q^{2-N}}{1+q}\left(\frac{1+q^2}{2}\right)^M z^2_{\frac{N+1}{2}}. \tag{2.10}$$

In this equation, the meaning of $D = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_N)$ as the signature of metric is clear, particularly in the limit $q \to 1$.

3. Contraction

We now apply the contraction procedure leading to the definition of the γ-Poincaré quantum group and the quantum spacetime on which it coacts. In the classical contraction scheme, the $(N-1)$-dimensional spacetime is identified with a neighbourhood of a particular point on the $N-1$ sphere (or hyperbola if the signature is Minkowskian), in the limit of infinite radius. Here we generalize this geometric point of view to non-commutative spaces. The two-dimensional situation was developed in detail in [12], both at the classical and quantum level. In particular, the geometric meaning of the different steps involved was made clear.

In the quantum space $O_q^N(\mathbb{R})$, we consider a subspace of dimension $N-1$ characterized by the condition $z^T\widetilde{C}z =$ constant (this corresponds to the de Sitter sphere in the classical Euclidean contraction). This subspace is invariant under the quantum group coaction because the quadratic form (2.10) is invariant. On this subspace, we select a particular point of coordinates $(z_i) = (R, 0, \ldots, 0)$ around which an expansion in R is performed. In the limit $R \to \infty$, this $(N-1)$-dimensional subspace will give rise to the quantum spacetime, and by a proper limit, the coaction (2.4) will induce a coaction of the γ-Poincaré quantum group.

We consider elements of $O_q^N(\mathbb{R})$ living on the subspace

$$z^T\widetilde{C}z = \epsilon_1\mathcal{R}^2. \tag{3.1}$$

We absorb an irrelevant factor in $R^2 = 2\mathcal{R}^2/1+q^{2-N}$. The factor ϵ_1 is compulsory if we want to keep all coordinates real when $R \to \infty$, as can be easily seen from (2.10) (recall also that in the contraction limit we choosed, $z_1 \to \infty$). The contraction amounts to take simultaneously $R \to \infty$ and $q \to 1$ by letting $q = \exp(\gamma/R)$, with γ a finite constant.

In (3.1) we choose to expand z_1 as a series in R (our convention for indices is that $i, j, k = 1, \ldots, N$, whereas $a, b, c = 2, \ldots, N$)

$$z_1 = R\left(1 - \frac{\epsilon_1}{2R^2}\sum_{a=2}^{N}\epsilon_a z_a^2 + \mathrm{O}(R^{-3})\right). \tag{3.2}$$

Inserting this expansion in the relations (2.9), the limit $R \to \infty$ is well defined because all the divergent terms cancel, and we are left with the unique constraint

$$[z_a, z_N] = -\mathrm{i}\gamma z_a. \tag{3.3}$$

We therefore define the quantum spacetime as the algebra generated by the z_a subject to the above constraint (3.3).

Next, we rewrite the generators of $Fun(SO_q(N,\mathbb{R}))$ as an expansion in the contraction parameter R

$$v_{ij} = \sum_{n=0}^{\infty} \frac{v_{ij}^n}{R^n} \tag{3.4}$$

and from simple requirements we will collect enough informations on the v_{ij}^n to enable us to derive all the necessary relations characterizing the algebra $Fun(\mathcal{P}_\gamma(N-1))$. First, we require that under the coaction δ of $Fun(SO_q(N,\mathbb{R}))$, the elements z_a remain of order 1 in the limit $R \to \infty$. Since

$$\delta(z_a) = v_{a1} \otimes z_1 + v_{ab} \otimes z_b$$

and z_1 is of order R, this is only possible if $v_{a1}^0 = 0$. Next we apply δ on both sides of (3.2) to get

$$\frac{1}{R}(v_{11} \otimes z_1 + v_{1a} \otimes z_a) = \mathbb{1} \otimes \mathbf{1} - \frac{\epsilon_1}{2R^2} \sum_{a=2}^{N} \epsilon_a \delta(z_a^2) + O(R^{-3}) \tag{3.5}$$

which implies that $v_{11}^0 = \mathbb{1}$ and $v_{11}^1 \otimes \mathbf{1} + v_{1a}^0 \otimes z_a = 0$, since $\delta(z_a^2)$ are finite by construction. As the elements $\mathbf{1}$ and z_a are linearly independent, we also conclude that $v_{1a}^0 = 0$ and $v_{11}^1 = 0$.

Collecting all this, we can take the $R \to \infty$ limit in $\delta(z) = V \dot{\otimes} z$, and dividing z_1 by R, this yields

$$\begin{aligned} &\delta(\mathbf{1}) = \mathbb{1} \otimes \mathbf{1} \\ &\delta(z_a) = v_{a1}^1 \otimes \mathbf{1} + v_{ab}^0 \otimes z_b. \end{aligned} \tag{3.6}$$

From this form of the coaction, we see that v_{a1}^1 play the role of translations and v_{ab}^0 the role of Lorentz transformations. It is then natural to take the elements $\mathbb{1}$, $u_{ab} = v_{ab}^0$ and $u_a = v_{a1}^1$ as the generators of $Fun(\mathcal{P}_\gamma(N-1))$, the algebra of functions on the quantum Poincaré group $\mathcal{P}_\gamma(N-1)$ (or, in short, the γ-Poincaré quantum group).

Now that we selected the generators of the algebra, we should determine the constraints imposed on them by the previous quantum group structure. First we apply the constraints that derive from the contraction of the two orthogonality relations (2.7). At zeroth order in $1/R$ one gets respectively

$$\begin{aligned} &v_{ab}^0 \epsilon_b v_{cb}^0 = \epsilon_a \delta_{ac} \\ &v_{ba}^0 \epsilon_b v_{bc}^0 = \epsilon_a \delta_{ac} \end{aligned} \tag{3.7}$$

and at first order, the relations are

$$\begin{aligned} &(v_{ji}^1 \epsilon_i v_{ki}^0 + v_{ji}^0 \epsilon_i v_{ki}^1 + \gamma v_{ji}^0 \epsilon_i \theta_i \rho_i v_{ki}^0 + \mathrm{i}\gamma v_{ji}^0 \epsilon_i \rho_i v_{ki'}^0) e_{jk} = \gamma \epsilon_i \theta_i \rho_i e_{ii} + \mathrm{i}\gamma \epsilon_i \rho_i e_{ii'} \\ &(v_{ij}^1 \epsilon_i v_{ik}^0 + v_{ij}^0 \epsilon_i v_{ik}^1 - \gamma v_{ij}^0 \epsilon_i \theta_i \rho_i v_{ik}^0 - \mathrm{i}\gamma v_{ij}^0 \epsilon_i \rho_i v_{i'k}^0) e_{jk} = -\gamma \epsilon_i \theta_i \rho_i e_{ii} - \mathrm{i}\gamma \epsilon_i \rho_i e_{ii'} \end{aligned} \tag{3.8}$$

where $\theta_i = 1$ if $i \leqslant M$ and $\theta_i = -1$ if $i > M$. These constraints will be useful when computing the antipode and the commutation relations.

The next task is to determine the commutation relations among the generators that derive from the contraction of the constraint $\mathcal{R}_v V_1 V_2 = V_2 V_1 \mathcal{R}_v$. For that purpose, one need to expand that expression up to order R^{-2} (in order to include the relations of v^1_{a1} with v^1_{b1}). Higher-order terms ($R^{-n}, n \geqslant 3$) will always contain elements v^n_{ij} of that order which by definition do not belong to the quantum Poincaré algebra, and thus do not yield new constraints on our set of generators. Performing the expansion, we get for the first three terms

$$[V^{(0)} \overset{\otimes}{,} V^{(0)}] = 0 \tag{3.9a}$$

$$[V^{(1)} \overset{\otimes}{,} V^{(0)}] + [V^{(0)} \overset{\otimes}{,} V^{(1)}] = [\mathcal{R}^{(1)}, V^{(0)} \otimes V^{(0)}] \tag{3.9b}$$

$$\begin{aligned}[V^{(1)} \overset{\otimes}{,} V^{(1)}] = &-[V^{(0)} \overset{\otimes}{,} V^{(2)}] - [V^{(2)} \overset{\otimes}{,} V^{(0)}] - [\mathcal{R}^{(2)}, V^{(0)} \otimes V^{(0)}] \\ &- \mathcal{R}^{(1)}(V_1^{(0)} V_2^{(1)} + V_1^{(1)} V_2^{(0)}) + (V_2^{(0)} V_1^{(1)} + V_2^{(1)} V_1^{(0)})\mathcal{R}^{(1)}.\end{aligned} \tag{3.9c}$$

We used the short-hand notation $X = \sum_n X^{(n)} R^{-n}$ for all the matrices, and the tensored commutator should be understood as $[V^{(n)} \overset{\otimes}{,} V^{(m)}]_{(ij,kl)} = [v^n_{ik}, v^m_{jl}]$.

Owing to the particular structure of $V^{(n)}$ obtained in (3.4)–(3.6), equation (3.9a) implies in components

$$[u_{ab}, u_{cd}] \equiv [v^0_{ab}, v^0_{cd}] = 0. \tag{3.10}$$

From equation (3.9b), we extract the commutation relation between the order zero and one generators of interest, v^0_{cd} and v^1_{a1}, namely we consider the component $(ac, 1d)$ of that equation. The necessary elements of the R-matrix are computed in appendix A, and one gets the commutation relations

$$[u_a, u_{cd}] \equiv [v^1_{a1}, v^0_{cd}] = \mathrm{i}\gamma\left((u_{Nd} - \delta_{Nd})\epsilon_1 \epsilon_a \delta_{ac} + (u_{cN} - \delta_{cN})u_{ad}\right). \tag{3.11}$$

From equation (3.9c), we determine the commutation relation between the order one generators, v^1_{a1}, considering the component $(ab, 11)$. This requires the knowledge of some particular matrix elements of $\mathcal{R}_v$ up to order R^{-2}, which can be found in appendix A. After some tedious but straightforward algebra, the result is

$$[u_a, u_b] \equiv [v^1_{a1}, v^1_{b1}] = \mathrm{i}\gamma(\delta_{Na} u_b - \delta_{Nb} u_a). \tag{3.12}$$

The other components of (3.9) are not relevant since they involves elements which are not part of the quantum Poincaré algebra as defined after (3.6).

The rest of the Hopf algebra structure is obtained by contracting the comultiplication $\Delta(V) = V \dot{\otimes} V$, which yields

$$\Delta(u_{ab}) = u_{ac} \otimes u_{cb} \qquad \Delta(u_a) = u_a \otimes \mathbb{1} + u_{ab} \otimes u_b \tag{3.13}$$

the counit $\epsilon(V) = \mathbb{1}$

$$\epsilon(u_{ab}) = \delta_{ab} \qquad \epsilon(u_a) = 0 \tag{3.14}$$

and the antipode (2.8)

$$\begin{aligned} S(u_{ab}) &= \epsilon_a \epsilon_b u_{ba} && \text{no sum on } a, b \\ S(u_a) &= -\epsilon_a u_{ba} \epsilon_b u_b && \text{no sum on } a. \end{aligned} \tag{3.15}$$

One can readily check that the commutation relations (3.10)–(3.12) satisfy the Jacobi identity and as they originate from a contraction of $Fun(SO_q(N,\mathbb{R}))$, it is natural to take them as the definition of the γ-Poincaré quantum group $Fun(\mathcal{P}_\gamma(N-1))$. Furthermore this definition is consistent with previous ones [3, 4, 11], obtained from quantization of a classical Poisson structure on the Poincaré group. It is remarkable that this deformation is linear in the deformation parameter γ. Here this linear dependence is a direct consequence of the contraction, where only the lowest order terms in the expansions are kept.

Looking closer at (3.7) and (3.10), one sees that $U = (u_{ab})$ actually describes an ordinary orthogonal matrix (with commuting entries) which preserves the metric $\eta_{ac} = \epsilon_1\epsilon_a\delta_{ac}$, i.e. equations (3.7) become $U^T\eta U = \eta$. The introduction of the factor ϵ_1 is suggested by (3.1), (3.2) and is natural when considering (3.11). In particular, because of the constraint (2.5) imposing $\epsilon_1 = \epsilon_N$, in our construction the time direction has always a positive signature $\eta_{NN} = 1$. One should mention that for odd-dimensional spacetime, this also forces the metric to have an odd/even number of plus/minus signs.

4. The γ-Poincaré quantum group as a bicrossproduct of algebra

It turns out that the Hopf algebra $Fun(\mathcal{P}_\gamma(N-1))$ just constructed by contraction can also be obtained as a bicrossproduct of two Hopf algebras, whose general theory was developed in [14]. Essentially, a bicrossproduct is a way to build a non-commutative non-cocommutative Hopf algebra from two Hopf algebras, using their respective (co)-actions on one another, provided some conditions are satisfied. Appendix B summarizes this construction.

In our case, the two algebras that form the bicrossproduct are the algebra of functions on the (classical) orthogonal group $Fun(SO(N-1,\mathbb{R}))$ and a non-commutative deformation of the algebra of translations T. The algebra $Fun(SO(N-1,\mathbb{R})) = A$ is generated as usual by the commuting elements $\overline{U} = (\overline{u}_{ab})$ (the indices $a, b, c, \ldots,$ take their $N-1$ values from 2 to N, in order to match with the notations in the previous sections) and has the Hopf algebra structure

$$\begin{aligned} &\Delta(\overline{U}) = \overline{U}\dot{\otimes}\overline{U} && \epsilon(\overline{U}) = \mathbb{1} \\ &S(\overline{U}) = \eta^{-1}\overline{U}^T\eta && \overline{U}^T\eta\overline{U} = \eta. \end{aligned} \tag{4.1}$$

Recall that $\eta = \mathrm{diag}(\epsilon_1\epsilon_2, \ldots, \epsilon_1\epsilon_N)$ and represents the metric in $\mathbb{R}^{N-1}$.

The translation algebra $T = H$ is generated by the elements $\overline{u}_a$ with the following relations:

$$\begin{aligned} &[\overline{u}_a, \overline{u}_b] = \mathrm{i}\gamma(\delta_{aN}\overline{u}_b - \delta_{bN}\overline{u}_a) && \epsilon(\overline{u}_a) = 0 \\ &\Delta(\overline{u}_a) = \overline{u}_a\otimes 1 + 1\otimes\overline{u}_a && S(\overline{u}_a) = -\overline{u}_a. \end{aligned} \tag{4.2}$$

For the bicrossproduct, we must define the actions and coactions. $Fun(SO)$ is a right T-module algebra with the structure map $\alpha : Fun(SO)\otimes T \to Fun(SO)$ given by

$$\alpha(\overline{u}_{ab}\otimes\overline{u}_c) \equiv \overline{u}_{ab}\triangleleft\overline{u}_c = \mathrm{i}\gamma\left((\overline{u}_{Nb} - \delta_{Nb})\eta_{ac} + (\overline{u}_{aN} - \delta_{aN})\overline{u}_{cb}\right). \tag{4.3}$$

We deliberately define an action that resembles (3.11) since, from the comultiplications in (4.1), (4.2) and the product (B.1) in the bicrossproduct, we see that this action eventually determines the commutator in K of elements of $Fun(SO)$ and T. One then verifies that it is a consistent action.

T is a left $Fun(SO)$-comodule coalgebra specified by the coaction $\beta : T \to Fun(SO)\otimes T$

$$\beta(\overline{u}_a) = \overline{u}_{ab}\otimes\overline{u}_b. \tag{4.4}$$

Here again, this expression is inspired by (3.13), since from (B.2) we see that β is the key ingredient for defining the comultiplication in K of an element of T. One can check that conditions (B.3) are satisfied by the structure maps (4.3), (4.4), therefore $K = T\bowtie Fun(SO)$ is a Hopf algebra. If in K we denote the elements

$$u_{ab} = \mathbf{1}\otimes\overline{u}_{ab} \qquad u_a = \overline{u}_a\otimes\mathbf{1}$$

and we apply the definitions for the product of appendix B, we get the following relations in K:

$$\begin{aligned}
&[u_{ab}, u_{cd}] = 0 \\
&[u_a\ , u_{cd}] = \mathrm{i}\gamma\,((u_{Nd} - \delta_{Nd})\eta_{ac} + (u_{cN} - \delta_{cN})u_{ad}) \\
&[u_a\ , u_b\ ] = \mathrm{i}\gamma\,(\delta_{Na}u_b - \delta_{Nb}u_a)
\end{aligned} \tag{4.5}$$

and for the comultiplication, counit and antipode

$$\begin{aligned}
&\Delta(u_{ab}) = u_{ac}\otimes u_{cb} \qquad \Delta(u_a) = u_a\otimes\mathbf{1} + u_{ab}\otimes u_b \\
&\epsilon(u_{ab}) = \delta_{ab} \qquad \epsilon(u_a) = 0 \\
&S(u_{ab}) = \eta_{ac}u_{dc}\eta_{db} \qquad S(u_a) = -\eta_{ab}u_{cb}\eta_{cd}u_d.
\end{aligned} \tag{4.6}$$

This shows that the bicrossproduct K is in fact the γ-Poincaré quantum group built in the previous section, $Fun(\mathcal{P}_\gamma(N-1)) = T\bowtie Fun(SO(N-1,\mathbb{R}))$.

Ultimately we would like to show that this γ-Poincaré quantum group is the Hopf algebra dual to the κ-Poincaré algebra of [1,2]. Since we established that the γ-Poincaré quantum group is a left–right bicrossproduct, a first step in that direction is to verify that the κ-Poincaré algebra is a right–left bicrossproduct. This is done in the next section for four-dimensional spacetime. To complete the duality proof, one should then prove that the action and coaction of the γ-Poincaré quantum group actually induce the coaction and action of the κ-Poincaré algebra. This is technically difficult in general, and for the time being we are able to perform it only for the two-dimensional case.

5. The κ-Poincaré algebra as a bicrossproduct

We explicitly construct the κ-Poincaré deformed algebra of [1,2] as a left–right bicrossproduct. A similar computation was carried out in [10], but for a right–left bicrossproduct. This minor difference comes from our choice of the opposite comultiplication for the κ-Poincaré algebra. This choice is arbitrary at the level of the Hopf algebra and is guided by our expressions for the γ-Poincaré quantum group. It has the inessential effect of permuting left and right in the bicrossproduct (co)-actions with respect to [10]. This time, the bicrossproduct combines a deformation of the algebra of translations with the enveloping algebra $U(so(3,1))$. The reduction to lower-dimensional spacetime or to other metric signature is straightforward.

$T^* = B$ is a non-cocommutative deformation of the enveloping algebra of translations with Hermitian generators $\mathcal{P}_\mu$ ($\mu, \nu = 0, 1, 2, 3$ and $r, s, t = 1, 2, 3$)

$$[\mathcal{P}_\mu, \mathcal{P}_\nu] = 0 \qquad \begin{cases} \Delta\mathcal{P}_0 = \mathcal{P}_0 \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{P}_0 \\ \Delta\mathcal{P}_r = \mathcal{P}_r \otimes \mathrm{e}^{-\mathcal{P}_0/\kappa} + \mathbf{1} \otimes \mathcal{P}_r. \end{cases} \tag{5.1}$$

$U(so(3,1)) = G$ is simply the enveloping algebra of the Lorentz Lie algebra, also with Hermitian generators

$$[M_r, M_s] = \mathrm{i}\epsilon_{rst} M_t \qquad [M_r, N_s] = \mathrm{i}\epsilon_{rst} N_t \qquad [N_r, N_s] = -\mathrm{i}\epsilon_{rst} M_t. \tag{5.2}$$

In the present situation as well, the definitions of the (co)-actions are guided by the final result we want to obtain. T^* is turned into a left $U(so)$-module algebra by the action

$$\begin{aligned} &M_r \triangleright \mathcal{P}_0 = 0 \qquad M_r \triangleright \mathcal{P}_s = \mathrm{i}\epsilon_{rst} \mathcal{P}_t \qquad N_r \triangleright \mathcal{P}_0 = i\mathcal{P}_r \\ &N_r \triangleright \mathcal{P}_s = \mathrm{i}\delta_{rs} \left(\frac{\kappa}{2}(1 - \mathrm{e}^{-2\mathcal{P}_0/\kappa}) + \frac{1}{2\kappa}\vec{\mathcal{P}}^2 \right) - \frac{\mathrm{i}}{\kappa} \mathcal{P}_r \mathcal{P}_s. \end{aligned} \tag{5.3}$$

$U(so)$ is a right T^*-comodule coalgebra with the coaction

$$\begin{aligned} &\delta(M_r) = M_r \otimes \mathbf{1} \\ &\delta(N_r) = N_r \otimes \mathrm{e}^{-\mathcal{P}_0/\kappa} - \frac{\mathrm{i}}{\kappa}\epsilon_{rst} M_s \otimes \mathcal{P}_t. \end{aligned} \tag{5.4}$$

These maps fulfill the conditions (B.6) and $L = T^* \blacktriangleright\!\!\!\triangleleft U(so(3,1))$ is a Hopf algebra. Putting

$$\hat{\mathcal{P}}_\mu = \mathcal{P}_\mu \otimes \mathbf{1} \qquad \hat{M}_r = \mathbf{1} \otimes M_r \qquad \hat{N}_r = \mathbf{1} \otimes N_r$$

one easily computes the following commutation relations in L:

$$\begin{aligned} &[\hat{\mathcal{P}}_\mu, \hat{\mathcal{P}}_\nu] = 0 \qquad [\hat{\mathcal{P}}_0, \hat{M}_r] = 0 \qquad [\hat{\mathcal{P}}_r, \hat{M}_s] = \mathrm{i}\epsilon_{rst}\hat{\mathcal{P}}_t \\ &[\hat{M}_r, \hat{M}_s] = \mathrm{i}\epsilon_{rst}\hat{M}_t \qquad [\hat{N}_r, \hat{N}_s] = -\mathrm{i}\epsilon_{rst}\hat{M}_t \\ &[\hat{N}_r, \hat{\mathcal{P}}_0] = i\hat{\mathcal{P}}_r \qquad [\hat{N}_r, \hat{\mathcal{P}}_s] = \mathrm{i}\delta_{rs}\left(\frac{\kappa}{2}(1 - \mathrm{e}^{-2\hat{\mathcal{P}}_0/\kappa}) + \frac{1}{2\kappa}\vec{\hat{\mathcal{P}}}^2\right) - \frac{\mathrm{i}}{\kappa}\hat{\mathcal{P}}_r\hat{\mathcal{P}}_s, \end{aligned} \tag{5.5}$$

and the comultiplications

$$\begin{aligned} &\Delta(\hat{\mathcal{P}}_0) = \hat{\mathcal{P}}_0 \otimes \mathbf{1} + \mathbf{1} \otimes \hat{\mathcal{P}}_0 \qquad \Delta(\hat{\mathcal{P}}_r) = \hat{\mathcal{P}}_r \otimes \mathrm{e}^{-\hat{\mathcal{P}}_0/\kappa} + \mathbf{1} \otimes \hat{\mathcal{P}}_r \\ &\Delta(\hat{M}_r) = \hat{M}_r \otimes \mathbf{1} + \mathbf{1} \otimes \hat{M}_r \\ &\Delta(\hat{N}_r) = \hat{N}_r \otimes \mathrm{e}^{-\hat{\mathcal{P}}_0/\kappa} + \mathbf{1} \otimes \hat{N}_r - \frac{\mathrm{i}}{\kappa}\epsilon_{rst}\hat{M}_s \otimes \hat{\mathcal{P}}_t. \end{aligned} \tag{5.6}$$

The antipode follows also easily. As expected, the relations (5.5) and (5.6) are those of the κ-Poincaré Hopf algebra [1, 2], in a somewhat different basis [10].

6. Duality in two dimensions

We will prove the duality of the (co)-actions in the case of the two-dimensional Minkowski γ-Poincaré quantum group. From (2.5), the restriction on the ϵ's imposes the signature $\epsilon_1 = \epsilon_3 = -1, \epsilon_2 = 1$, therefore $\eta_{33} = 1 = -\eta_{22}$. All the calculations below can be extended to the Euclidean situation without difficulty.

In the Lorentz group, it is more convenient to describe a boost by the rapidity parameter

$$\overline{U} = (\overline{u}_{ab}) = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \tag{6.1}$$

and to take θ as the generator of $Fun(SO(1,1))$.

The algebra T is generated by $\overline{u}_2, \overline{u}_3$ constrained by the commutator $[\overline{u}_3, \overline{u}_2] = \mathrm{i}\gamma\overline{u}_2$ and a basis of it is given by the ordered monomials $\overline{u}_2^n\overline{u}_3^m$ for non-negative integers n, m. From (4.3) and (6.1), the action of T on $Fun(SO(1,1))$ is

$$\theta \triangleleft \overline{u}_2 = \mathrm{i}\gamma(\mathbf{1} - \cosh\theta)$$

$$\theta \triangleleft \overline{u}_3 = -\mathrm{i}\gamma \sinh\theta$$

and the coaction (4.4) is

$$\beta(\overline{u}_2) = \cosh\theta \otimes \overline{u}_2 + \sinh\theta \otimes \overline{u}_3$$

$$\beta(\overline{u}_3) = \sinh\theta \otimes \overline{u}_2 + \cosh\theta \otimes \overline{u}_3.$$

In T^* we take the commuting generators P_2, P_3 with the pairing

$$\left.\begin{aligned} \langle \overline{u}_2^n\overline{u}_3^m, P_2\rangle &= \delta_{n,1}\delta_{m,0} \\ \langle \overline{u}_2^n\overline{u}_3^m, P_3\rangle &= \delta_{n,0}\delta_{m,1} \end{aligned}\right. \quad \Rightarrow \quad \langle \overline{u}_2^n\overline{u}_3^m, P_2^p P_3^q\rangle = n!m!\delta_{n,p}\delta_{m,q}$$

and from the $\overline{u}_a$'s commutation relation we deduce their comultiplication

$$\Delta(P_2) = P_2 \otimes \mathbf{1} + \mathrm{e}^{\mathrm{i}\gamma P_3} \otimes P_2$$

$$\Delta(P_3) = P_3 \otimes \mathbf{1} + \mathbf{1} \otimes P_3.$$

In $Fun(SO(1,1))^* = U(so(1,1))$ we single out the generator N with the pairing

$$\langle \theta^n, N\rangle = \delta_{n,1}.$$

Since $Fun(SO(1,1))$ is a right T-module, $U(so(1,1))$ is a right T^*-comodule. The most general coaction is

$$\delta(N) = c_{p,nm} N^p \otimes P_2^n P_3^m.$$

To compute the coefficients, we use the duality relation

$$\langle \theta^p \triangleleft \overline{u}_2^n\overline{u}_3^m, N\rangle = \langle \theta^p \otimes \overline{u}_2^n\overline{u}_3^m, \delta(N)\rangle = p!n!m!c_{p,nm}.$$

From

$$p!c_{p,1\,0} = \langle \theta^p \triangleleft \overline{u}_2, N\rangle = \langle p\theta^{p-1} \mathrm{i}\gamma(1-\cosh\theta), N\rangle = 0$$

we deduce that

$$p!n!c_{p,n\,0} = \langle \theta^p \triangleleft \overline{u}_2^n, N\rangle = \langle (\theta^p \triangleleft \overline{u}_2^{n-1}) \otimes \overline{u}_2, \delta(N)\rangle = 0. \tag{6.2}$$

Similarly, one easily finds that $c_{p,0\,1} = -\mathrm{i}\gamma\delta_{p,1}$, which allows one to establish the recurrence relation

$$p!m!c_{p,0\,m} = \langle (\theta^p \triangleleft \overline{u}_3^{m-1}) \otimes \overline{u}_3, \delta(N)\rangle = -\mathrm{i}\gamma\langle \theta^p \triangleleft \overline{u}_3^{m-1}, N\rangle = -\mathrm{i}\gamma p!(m-1)!c_{p,0\,m-1}$$

solved by

$$c_{p,0\,m} = \frac{(-\mathrm{i}\gamma)^m}{m!}\delta_{p,1}. \tag{6.3}$$

The coefficients for strictly positive n, m vanish since

$$\langle \theta^p \triangleleft \overline{u}_2^n \overline{u}_3^m, N\rangle = \langle (\theta^p \triangleleft \overline{u}_2^n) \otimes \overline{u}_3^m, \delta(N)\rangle = \langle \theta^p \triangleleft \overline{u}_2^n, N\rangle \langle \overline{u}_3^m, \mathrm{e}^{-\mathrm{i}\gamma P_3}\rangle = 0$$

as a consequence of (6.2) and (6.3). Therefore the coaction is

$$\delta(N) = N \otimes \mathrm{e}^{-\mathrm{i}\gamma P_3}. \tag{6.4}$$

As T is a left $Fun(SO(1,1))$-comodule, T^* is a left $U(so(1,1))$-module, and we have to compute

$$N \triangleright P_a = d_{a,nm} P_2^n P_3^m$$

using the duality

$$\langle \overline{u}_2^n \overline{u}_3^m, N \triangleright P_a\rangle = \langle \beta(\overline{u}_2^n \overline{u}_3^m), N \otimes P_a\rangle = n!m!d_{a,nm}.$$

For U any element of T^*, we have from (B.1)

$$\beta(U\overline{u}_a) = U^{\overline{1}} \triangleleft \overline{u}_a \otimes U^{\overline{2}} + U^{\overline{1}}\overline{u}_{ac} \otimes U^{\overline{2}}\overline{u}_c. \tag{6.5}$$

Therefore, using the coaction (6.4), we get

$$\langle \beta(U\overline{u}_a), N \otimes P_b\rangle = -\mathrm{i}\gamma\delta_{a,3}\langle \beta(U), N \otimes P_b\rangle + \langle U^{\overline{1}} u_{ac}, N\rangle \langle U^{\overline{2}} \otimes u_c, \Delta(P_b)\rangle. \tag{6.6}$$

For $b = 3$, the second term always vanishes except when $U = 1$ and $a = 2$ and we find

$$\langle \beta(\overline{u}_2^n \overline{u}_3^m), N \otimes P_3\rangle = (-\mathrm{i}\gamma)^m \delta_{n,1} = n!m!d_{3,nm}$$

which yields

$$N \triangleright P_3 = P_2 \mathrm{e}^{-\mathrm{i}\gamma P_3}. \tag{6.7}$$

When the index $b=2$, equation (6.6) reduces to

$$\langle\beta(U\overline{u}_a), N\otimes P_2\rangle = \delta_{a,3}\left(-\mathrm{i}\gamma\langle\beta(U), N\otimes P_2\rangle + \langle\beta(U), 1\otimes \mathrm{e}^{\mathrm{i}\gamma P_3}\rangle\right) + \delta_{a,2}\langle\beta(U), N\otimes \mathrm{e}^{\mathrm{i}\gamma P_3}\rangle.$$

Using the pairings

$$\langle\beta(\overline{u}_2^n\overline{u}_3^m), \mathbf{1}\otimes \mathrm{e}^{\mathrm{i}\gamma P_3}\rangle = (\mathrm{i}\gamma)^m\delta_{n,0}$$

$$\langle\beta(\overline{u}_2^n\overline{u}_3^m), N\otimes \mathrm{e}^{\mathrm{i}\gamma P_3}\rangle = \mathrm{i}\gamma\delta_{n,1}\delta_{m,0}$$

we get, for $n>0$

$$\langle\beta(\overline{u}_2^n\overline{u}_3^m), N\otimes P_2\rangle = (-\mathrm{i}\gamma)^m(\mathrm{i}\gamma)\delta_{n-1,1} = n!m!d_{2,nm}$$

and for $n=0$

$$\langle\beta(\overline{u}_3^m), N\otimes P_2\rangle = -\mathrm{i}\gamma\langle\beta(\overline{u}_3^{m-1}), N\otimes P_2\rangle + (\mathrm{i}\gamma)^{m-1}.$$

This last recurrence is solved by the coefficients

$$d_{2,0\,2m} = 0 \qquad d_{2,0\,2m+1} = \frac{1}{\mathrm{i}\gamma}\frac{(\mathrm{i}\gamma)^{2m+1}}{(2m+1)!}$$

and we finally get the action

$$N \triangleright P_2 = \frac{1}{\mathrm{i}\gamma}\sinh(\mathrm{i}\gamma P_3) + \frac{\mathrm{i}\gamma}{2}P_2^2\mathrm{e}^{-\mathrm{i}\gamma P_3}. \tag{6.8}$$

Before making contact with the previous section, we should be careful about the Hermitian properties of the generators N, P_a. Knowing that $\theta, \overline{u}_a$ are Hermitian, these are established using the definition (see [15] for example)

$$\langle\overline{u}_2^n\overline{u}_3^m, P_a^*\rangle = \overline{\langle S(\overline{u}_2^n\overline{u}_3^m)^*, P_a\rangle} \qquad \langle\theta^n, N^*\rangle = \overline{\langle S(\theta^n)^*, N\rangle}$$

and we find that N, P_a are actually anti-Hermitian.

If we define the new Hermitian generators $\mathcal{N} = \mathrm{i}N, \mathcal{P}_2 = \mathrm{i}P_2\mathrm{e}^{-\mathrm{i}\gamma P_3}, \mathcal{P}_3 = \mathrm{i}P_3$, equations (6.4), (6.7) and (6.8) become

$$\begin{aligned}
&\delta(\mathcal{N}) = \mathcal{N}\otimes \mathrm{e}^{-\gamma\mathcal{P}_3}\\
&\mathcal{N}\triangleright\mathcal{P}_3 = \mathrm{i}\mathcal{P}_2\\
&\mathcal{N}\triangleright\mathcal{P}_2 = \frac{\mathrm{i}}{2\gamma}\left(\mathbf{1} - \mathrm{e}^{-2\gamma\mathcal{P}_3}\right) - \frac{\mathrm{i}\gamma}{2}\mathcal{P}_2^2
\end{aligned} \tag{6.9}$$

which is clearly the reduction of the maps (5.3) and (5.4) to the two-dimensional situation, with the substitution $\kappa = 1/\gamma$. Therefore the two-dimensional κ-Poincaré algebra is the bicrossproduct dual to the γ-Poincaré quantum group.

7. Conclusion

There are good reasons to believe that the γ-Poincaré quantum group is in fact the dual to the κ-Poincaré Hopf algebra. The approach proposed here is very reminiscent of the contraction used in deriving the κ-Poincaré algebra: we start from a dual structure and the deformation parameter q is treated in the same way. Furthermore, the bicrossproduct formulations of these two Hopf algebras appears to be dual to each other, as the two-dimensional proof of section 6 shows.

The advantage of using the bicrossproduct structure of the γ-Poincaré quantum group and algebra is that they are split into their building blocks which are easier to handle, being simpler mathematical structures. The algebra of functions on the classical Lorentz group is dual to the enveloping algebra of the Lorentz Lie algebra [16] and obviously T^* is dual to T. Therefore, as vector spaces, the κ-Poincaré algebra and the γ-Poincaré quantum group are dual. Showing that the algebraic structures on these spaces are dual reduces to the proof of the (co)-actions duality.

The difficulty in generalizing the result of section 6 to higher dimensions lies mainly in the definition of dual basis. The presentation of the bicrossproducts are simpler in the respective bases (4.1)–(4.4) and (5.1)–(5.4), but these are very inconvenient bases when dealing with the duality issue.

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Appendix A. The R-matrix

In order to derive the commutation relations (3.9) of the γ-Poincaré quantum group by contraction, we need to expand the R-matrix (2.1) up to second order in R. First, one has to express it in the v_{ij} basis, using the matrix M (2.6)

$$\mathcal{R}_v = M\otimes M\ \mathcal{R}_t\ M^{-1}\otimes M^{-1} = \sum_{n=0}^{\infty} \mathcal{R}_v^{(n)} R^{-n}.$$

It is obvious that the zeroth-order term $\mathcal{R}_v^{(0)}$ is the identity matrix, and this explains the simplicity of the result (3.10).

The first-order term $\mathcal{R}_v^{(1)}$ can be recast after some algebra in the conventional form

$$\mathcal{R}_v^{(1)} = \gamma \sum_{i=1}^{M} H_i\otimes H_i + 2\gamma \sum_{\alpha\in\Delta_+} E_{-\alpha}\otimes E_\alpha. \tag{A.1}$$

In this equation, $H_i, E_{\pm\alpha}$ are the Cartan–Weyl generators of the Lie algebra $so(N,\epsilon)$ in the defining representation. Given an orthonormal basis e_i, $1\leqslant i\leqslant M$ of $\mathbb{R}^M$, the positive (long) roots Δ_+ are $e_i \pm e_j$, $1\leqslant i<j\leqslant M$, and when N is odd, the additional positive short roots are e_i, $1\leqslant i\leqslant M$. Putting

$$N_{ij} = -\mathrm{i}\epsilon_i e_{ij} + \mathrm{i}\epsilon_j e_{ji}$$

these generators are

$$H_i = \epsilon_i N_{ii'}$$

$$E_{e_i\pm e_j} = \tfrac{1}{2}(N_{ij} + \mathrm{i}N_{i'j} \pm \mathrm{i}N_{ij'} \mp N_{i'j'}) \qquad E_{e_i} = \frac{1}{\sqrt{2}}(N_{i\frac{N+1}{2}} + \mathrm{i}N_{i'\frac{N+1}{2}}) \tag{A.2}$$

$$E_{-(e_i\pm e_j)} = \frac{\epsilon_i\epsilon_j}{2}(N_{ij} - \mathrm{i}N_{i'j} \mp \mathrm{i}N_{ij'} \mp N_{i'j'}) \qquad E_{-e_i} = \frac{\epsilon_i}{\sqrt{2}}(N_{i\frac{N+1}{2}} - \mathrm{i}N_{i'\frac{N+1}{2}}).$$

For the second commutator $[v^1_{a1}, v^0_{cd}]$ one first remarks that due to the structure of the matrices $V^{(0,1)}$, the term $[V^{(0)} \overset{\otimes}{,} V^{(1)}]_{(ac,1d)}$ vanishes, and the right-hand side is

$$[\mathcal{R}^{(1)}, V^{(0)} \otimes V^{(0)}]_{(ac,1d)} = \mathcal{R}^{(1)}_{(ac,1b)} v^0_{bd} - v^0_{ab} v^0_{ce} \mathcal{R}^{(1)}_{(be,1d)}.$$

From (A.1) and (A.2) one computes the relevant matrix element

$$\mathcal{R}^{(1)}_{(ab,1c)} = \mathrm{i}\gamma(\delta_{Nb}\delta_{ac} - \epsilon_1\epsilon_a\delta_{Nc}\delta_{ab})$$

and we get

$$[\mathcal{R}^{(1)}, V^{(0)} \otimes V^{(0)}]_{(ac,1d)} = -\mathrm{i}\gamma\left((v^0_{Nd} - \delta_{Nd})\epsilon_1\epsilon_a\delta_{ac} + (v^0_{cN} - \delta_{cN})v^0_{ad}\right).$$

For the last commutator $[v^1_{a1}, v^1_{b1}]$, the terms $[V^{(0)} \overset{\otimes}{,} V^{(2)}]_{(ab,11)}$ and $[V^{(2)} \overset{\otimes}{,} V^{(0)}]_{(ab,11)}$ vanish. For the rest, we need the linear and quadratic terms in $\mathcal{R}_v$ and in particular

$$\left(\mathcal{R}^{(1)}(V_1^{(0)}V_2^{(1)} + V_1^{(1)}V_2^{(0)}) + (V_2^{(0)}V_1^{(1)} + V_2^{(1)}V_1^{(0)})\mathcal{R}^{(1)}\right)_{(ab,11)}. \tag{A.3}$$

Again, due to the specific structure of the $V^{(0,1)}$ matrices, only some additional matrix elements enter the above equation and are found to be

$$\mathcal{R}^{(1)}_{(ab,c1)} = -\mathrm{i}\gamma(\delta_{Na}\delta_{bc} - \epsilon_1\epsilon_a\delta_{Nc}\delta_{ab})$$

$$\mathcal{R}^{(1)}_{(ic,11)} = -\gamma\epsilon_1\epsilon_i\delta_{ic} = \mathcal{R}^{(1)}_{(ci,11)}.$$

When inserted in (A.3), one gets

$$\begin{aligned} &\left(\mathcal{R}^{(1)}(V_1^{(0)}V_2^{(1)} + V_1^{(1)}V_2^{(0)})\right)_{(ab,11)} = \mathrm{i}\gamma(\delta_{Nb}v^1_{a1} - \delta_{Na}v^1_{b1}) \\ &\left((V_2^{(0)}V_1^{(1)} + V_2^{(1)}V_1^{(0)})\mathcal{R}^{(1)}\right)_{(ab,11)} = -\gamma\epsilon_1\epsilon_c(v^0_{bc}v^1_{ac} + v^1_{bc}v^0_{ac}). \end{aligned} \tag{A.4}$$

The last contribution to (3.9c) is

$$[\mathcal{R}^{(2)}, V^{(0)} \otimes V^{(0)}]_{(ab,11)} = \mathcal{R}^{(2)}_{ab,11} - v^0_{ac}v^0_{bd}\mathcal{R}^{(2)}_{cd,11}. \tag{A.5}$$

Expanding the R-matrix up to order two, one gets

$$\mathcal{R}^{(2)}_{ab,11} = \gamma^2(-\epsilon_1\epsilon_a\theta_a\rho_a\delta_{ab} - \mathrm{i}\epsilon_1\epsilon_b\rho_b\delta_{ab'} + 2\epsilon_1\epsilon_a\rho_1\delta_{ab}).$$

Therefore equation (A.5) becomes (sum on c only)

$$\begin{aligned} [\mathcal{R}^{(2)}, V^{(0)} \otimes V^{(0)}]_{(ab,11)} &= \gamma^2\epsilon_1(v^0_{bc}\epsilon_c\theta_c\rho_c v^0_{ac} + \mathrm{i}v^0_{bc}\epsilon_c\rho_c v^0_{ac'} - \epsilon_a\theta_a\rho_a\delta_{ab} - \mathrm{i}\epsilon_b\rho_b\delta_{ab'}) \\ &= -\gamma\epsilon_1\epsilon_c(v^0_{bc}v^1_{ac} + v^1_{bc}v^0_{ac}). \end{aligned}$$

In the last step, we used the first orthogonality relation (3.8) (putting $j = b, k = a$) in order to simplify the expression, and one sees that it cancels with the second contribution in (A.4), leaving the result (3.12).

Appendix B. The bicrossproduct

In this appendix, we recall the bicrossproduct construction of Majid, setting up the notation used in the main text. Some detailed proofs can be found in [14].

Let H, A be two Hopf algebras, where A is a right H-module algebra with the structure map $\alpha : A\otimes H \to A$,

$$\alpha(a\otimes h) = a\triangleleft h \qquad h \in H, a \in A$$

and H is a left A-comodule coalgebra with the structure map $\beta : H \to A\otimes H$

$$\beta(h) = h^{\overline{1}} \otimes h^{\overline{2}} \qquad h, h^{\overline{2}} \in H, h^{\overline{1}} \in A.$$

On the smash product-coproduct $K = H\#A$ (which is isomorphic to $H\otimes A$ as a vector space) one can put both a structure of algebra with the multiplication rule

$$(h\otimes a)\cdot(g\otimes b) = hg_{(1)}\otimes(a\triangleleft g_{(2)})b \tag{B.1}$$

and a structure of coalgebra with the comultiplication

$$\Delta(h\otimes a) = h_{(1)}\otimes {h_{(2)}}^{\overline{1}}a_{(1)}\otimes {h_{(2)}}^{\overline{2}}\otimes a_{(2)}. \tag{B.2}$$

The comultiplication is denoted by $\Delta(h) = h_{(1)}\otimes h_{(2)}$. K is a bialgebra if and only if

$$\begin{aligned}
&\epsilon(a\triangleleft h) = \epsilon(a)\epsilon(h) \qquad \text{and} \qquad \beta(1) = 1\otimes 1\\
&\Delta(a\triangleleft h) = (a_{(1)}\triangleleft h_{(1)}){h_{(2)}}^{\overline{1}}\otimes a_{(2)}\triangleleft {h_{(2)}}^{\overline{2}}\\
&\beta(hg) = (h^{\overline{1}}\triangleleft g_{(1)}){g_{(2)}}^{\overline{1}}\otimes h^{\overline{2}}{g_{(2)}}^{\overline{2}}\\
&{h_{(1)}}^{\overline{1}}(a\triangleleft h_{(2)})\otimes {h_{(1)}}^{\overline{2}} = (a\triangleleft h_{(1)}){h_{(2)}}^{\overline{1}}\otimes {h_{(2)}}^{\overline{2}}.
\end{aligned} \tag{B.3}$$

These conditions arise from the compatibility of the multiplication and the comultiplication in K. Then K is even a Hopf algebra with the antipode

$$S(h\otimes a) = (1\otimes S(h^{\overline{1}}a))\cdot(S(h^{\overline{2}})\otimes 1).$$

K is called a right–left bicrossproduct and is denoted by $H\blacktriangleright\!\!\triangleleft A$.

This structure has a dual counterpart, where left and right are exchanged. This time let B be a left G-module algebra with the structure map $\gamma : G\otimes B \to B$

$$\gamma(g\otimes b) = g\triangleright b \qquad g \in G, b \in B$$

and G be a right B-comodule coalgebra with the structure map $\delta : G \to G\otimes B$

$$\delta(g) = g^{\overline{1}} \otimes g^{\overline{2}} \qquad g, g^{\overline{1}} \in G, g^{\overline{2}} \in B.$$

On the smash product–coproduct $L = B\#G$ the multiplication rule is

$$(a\otimes h)\cdot(b\otimes g) = a(h_{(1)}\triangleright b)\otimes h_{(2)}g \tag{B.4}$$

and the comultiplication

$$\Delta(b\otimes g) = b_{(1)}\otimes g_{(1)}{}^{\bar{1}}\otimes b_{(2)}g_{(1)}{}^{\bar{2}}\otimes g_{(2)}. \tag{B.5}$$

L is a bialgebra iff

$$\begin{aligned}
&\epsilon(g\triangleright b) = \epsilon(g)\epsilon(b) \qquad \text{and} \qquad \delta(1) = 1\otimes 1 \\
&\Delta(g\triangleright b) = g_{(1)}{}^{\bar{1}}\triangleright b_{(1)}\otimes g_{(1)}{}^{\bar{2}}(g_{(2)}\triangleright b_{(2)}) \\
&\delta(gh) = g_{(1)}{}^{\bar{1}}h^{\bar{1}}\otimes g_{(1)}{}^{\bar{2}}(g_{(2)}\triangleright h^{\bar{2}}) \\
&g_{(2)}{}^{\bar{1}}\otimes(g_{(1)}\triangleright b)g_{(2)}{}^{\bar{2}} = g_{(1)}{}^{\bar{1}}\otimes g_{(1)}{}^{\bar{2}}(g_{(2)}\triangleright b).
\end{aligned} \tag{B.6}$$

Then L is a Hopf algebra with antipode

$$S(b\otimes g) = (1\otimes S(g^{\bar{1}}))\cdot(S(bg^{\bar{2}})\otimes 1)$$

and is called a left–right bicrossproduct denoted by $B\blacktriangleright\!\!\triangleleft G$.

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